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Quark-Gluon Amplitudes in the Dual Expansion

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Abstract

The dual representation, which gives a simple analytical form for purely gluonic amplitudes, is extended to amplitudes which include a quark-antiquark pair. To minimize the calculations, supersymmetry is used to relate the purely gluonic amplitudes to those including a gluino pair from which the quark-antiquark amplitudes are easily deduced. We explicitly give simple analytical forms for the full amplitudes for those multi-parton processes which involve a quark-antiquark pair plus two, three and four gluons.

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1 Introduction

Recently a new technique has been developed to calculate multi-gluon amplitudes [1,2]. The key element of this technique is the use of a *Chan-Paton* basis[3] for the color structure of the gluonic amplitudes. For each Feynman diagram contributing to a given process the color factor is rewritten in terms of traces of products of $SU(N)$ matrices in the fundamental representation N. Then the diagrams containing traces of $SU(N)$ matrices in the same ordering are summed together, giving the dual ***sub-amplitudes*** and the full amplitude is the sum over non-cyclic permutations of the external gluons of these sub-amplitudes multiplied by the trace of the $SU(N)$ matrices in the given permutation.

The suggestion that this color basis might be particularly useful comes from the analogy with the dual models, where amplitudes among massless gauge vectors are expressed in exactly this way. For this reason we also refer to the Chan-Paton basis as the *dual basis.* In references [1,2] the efficacy of this technique was explicitly shown through the analytic calculation of the six-gluon matrix element. At first sight, however, this technique does not readily extend to the study of processes involving quarks. We certainly know that the color structure of diagrams involving quarks does not admit a Chan-Paton representation. Also, dual models describing quarks and gluons do not exist, even at tree level where constraints like the critical dimension of the space-time do not apply.

Scattering processes involving a quark-antiquark pair are very important for today's and future hadronic colliders[4]. First of all they contribute to a substantial part of the jet cross-section. Secondly, they give rise to interesting processes like production of weak bosons and of heavy flavours. This calls for a formalism in which these phenomena can be described by relatively simple formulae. A generalization of the technique used in [1,2] to study the gluonic processes seems to be the best candidate for such a formalism. In spite of the obstacles pointed out before, in reference [5] we were able to construct a set of helicity amplitudes containing gluons and quarks by exploiting the factorization properties of the gluon sub-amplitudes. In this paper we extend that construction, and we give a full recipe to handle amplitudes containing a quark-antiquark pair in a spirit very close to that of the dual expansion for the gluon-only processes. We also show how to use this technique in the study of more exotic processes, like the production of supersymmetric particles (gluinos and scalar quarks).

In order to limit the amount of calculations, we complement the dual expansion with the *supersymmetry trick*, connecting amplitudes with external states obeying different statistics through Supersymmetry Ward Identities (SWI's). A SWI acting on the full amplitude for a given gluonic process gives rise to identities among processes involving gluinos, *i.e.* fermions transforming under the adjoint of $SU(N)$. This allows us to calculate gluino amplitudes from gluon amplitudes, or vice versa, without having to evaluate any new sets of Feynman diagrams. We will show how to relate gluino amplitudes to quark amplitudes and will implement these techniques by explicitly calculating the full matrix element for the two quark-four gluon process, finding a compact analytic expression for it.

This paper is organized as follows. In Section 2 we review the supersymmetry trick. In Section 3 we review the dual expansion technique and we study the color structure of diagrams involving a quark-antiquark pair, finding a prescription to generalize the dual basis to the case of particles transforming in a representation other than the adjoint. In Section 4 we recall the formulae describing the five- and six-gluon matrix element and give the results for the $(\bar{q}qggg)$ and $(\bar{q}qgggg)$ processes. We will collect some of the details of the derivation in an Appendix. In section 5 we will briefly analyse some amplitudes describing rarer processes, like $(n > 6)$-parton scattering and supersymmetric particle production in multi-jet events. We hope that these last examples will convince the reader of the power and generality of this technique. In Section 6 we will present our conclusions.

2 The Supersymmetry Trick

The use of Supersymmetry Ward Identities (SWI's) for the calculation of tree level gluon scattering in QCD was first proposed in [6]. Supersymmetry transforms bosons into fermions and vice versa. In our analysis here we will study simple $N = 1$ supersymmetry, instead of the extended $N = 2$ supersymmetry employed in [6,7]. $N = 1$ supersymmetry was already used in reference [8] for the calculation of the six-parton scattering.

One possible representation of $N = 1$ supersymmetry contains a massless vector $(g^{\pm})$ and a massless spin 1/2 Weyl spinor $(\Lambda^{\pm})$. The $\pm$ refers to the two possible helicity states of the vector and the spinor. Let $Q(\eta)$ be the supersymmetry charge with η being the fermionic parameter of the transformation. Then $Q(\eta)$ acts on the doublet (g, Λ) as follows[10]:

$$[Q(\eta), g^{\pm}(p)] = \mp \Gamma^{\pm}(p,\eta)\ \Lambda^{\pm}, \tag{2.1}$$

$$[Q(\eta), \Lambda^{\pm}(p)] = \mp \Gamma^{\mp}(p,\eta)\ g^{\pm}. \tag{2.2}$$

$\Gamma^{\pm}(p,\eta)$ is a complex function linear in η and satisfies:

$$\Gamma^{+}(p,\eta) = [\Gamma^{-}(p,\eta)]^{*} = \bar{\eta}\ u^{(-)}(p), \tag{2.3}$$

with $u^{(-)}(p)$ a negative helicity spinor satisfying the massless Dirac equation with momentum p. Because of the arbitrariness in choosing the supersymmetry parameter η, we choose this to be a negative helicity spinor obeying the Dirac equation with an arbitrary massless momentum k. Here and in the rest of the paper we adopt the Xu *et al.* *improved* version[11] of the CALKUL parametrization for the helicities[12]. In Appendix A we collect the relevant definitions. According to these definitions we can then write:

$$\Gamma^+(p,k) \equiv \Gamma^+(p,\eta(k)) = \langle k+|p-\rangle \equiv [kp]. \tag{2.4}$$

As a notation, we choose to label the supersymmetry charge $Q(\eta)$ with the momentum k characterising the parameter η: $Q(k) = Q[\eta(k)]$.

Because of supersymmetry, the operator $Q(k)$ annihilates the vacuum. It follows that the commutator of $Q(k)$ with any string of operators creating or annihilating vectors $g^{\pm}$ and spinors $\Lambda^{\pm}$ has a vanishing vacuum expectation value. If z_i represents any of these operators, we then obtain the following identity[10]:

$$0 = \langle [Q, \prod_{i=1}^{n} z_i] \rangle_0 = \sum_{i=1}^{n} \langle z_1 \cdots [Q, z_i] \cdots z_n \rangle_0, \tag{2.5}$$

where $\langle \ldots \rangle_0$ indicates the vacuum expectation value. If we substitute in equation (2.5) the commutators, we obtain a relation among scattering amplitudes for particles with different spin. The amplitudes with only vectors are the same that one would have in the ordinary non-supersymmetric theory, because at tree level the supersymmetric partners of the vectors do not appear as intermediate states. General features of Yang-Mills interactions, like helicity conservation in the fermion-fermion-vector vertex guarantee the vanishing of some of the amplitudes in (2.5). The arbitrariness in choosing the reference momentum k for the supersymmetry parameter η allows a further simplification of equation (2.5), by choosing k to be equal to one of the external momenta.

To be more explicit, let us give an example. Consider the helicity amplitude $(g_1^-, g_2^-, g_3^+, \ldots, g_n^+)$, with two negative-helicity gluons and $n-2$ positive-helicity gluons where all of the particles are outgoing. Through the SWI we can relate this amplitude to amplitudes with two gluinos and $n-2$ gluons. Helicity conservation for the fermions implies that only an amplitude with one positive- and one negative-helicity gluino can be non-vanishing. In this way equation (2.5) reduces to:

$$\begin{aligned} \Gamma^-(p_1,k)A(\Lambda_1^-, g_2^-, \Lambda_3^+, g_4^+, \ldots, g_n^+) \quad &+ \quad \Gamma^-(p_2,k)A(g_1^-, \Lambda_2^-, \Lambda_3^+, g_4^+, \ldots, g_n^+) \\ &- \quad \Gamma^-(p_3,k)A(g_1^-, g_2^-, g_3^+, \ldots, g_n^+) = 0. \end{aligned} \tag{2.6}$$

As we said before, the purely gluonic amplitude for the non-supersymmetric and the supersymmetric theory coincide. The relevant gluon amplitude was given in [1,2]:

$$A(g_1^-, g_2^-, g_3^+, \ldots, g_n^+) = ig^{n-2}\langle 12\rangle^4 \sum_{perm'} tr(\lambda_1\lambda_2\cdots\lambda_n)\frac{1}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}, \tag{2.7}$$

where the sum is taken over the $(n-1)!$ non-cyclic permutations on the indices. If we then choose $k = p_1$, we obtain the following relation:

$$A(g_1^-, \Lambda_2^-, \Lambda_3^+, g_4^+, \ldots, g_n^+) = ig^{n-2}\langle 12\rangle^3\langle 23\rangle \sum_{perm'} tr(\lambda_1\lambda_2\cdots\lambda_n)\frac{1}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}. \tag{2.8}$$

For $n = 5$ this agrees with a previously known expression[6].

3 The Color Structure and the Dual Basis

In references [1,2] the suggestion was made to expand any n-gluon matrix element into a Chan-Paton color basis[3]. In this basis all of the color factors are expressed through traces of $SU(N)$ matrices in the fundamental representation (λ matrices), and the full amplitude is given by the following expression:

$$A_n = \sum_{perm'} tr\,(\lambda_1\lambda_2\ldots\lambda_n)\; m(1,2,\ldots,n). \tag{3.1}$$

We will call the functions $m(1,2,\ldots,n)$ *sub-amplitudes*. They are only functions of the kinematical variables of the process, *i.e.* the momenta and the helicities of the external gluons. These variables are represented with a shorthand notation by the indices $(1,2,\ldots,n)$. We will use the symbol $A(\ldots)$ for the *full* amplitude, where the ordering of the indices is irrelevant. The sum in Equation (3.1) is taken over all the $(n-1)!$ *non-cyclic* permutations of the indices $1,2,\ldots,n$. We will normalize the λ matrices so that $[\lambda_a, \lambda_b] = i\sqrt{2}f_{abc}\lambda_c$ and $tr(\lambda_a\lambda_b) = \delta_{ab}$.

The sub-amplitudes satisfy many important properties:

1. $m(1,2,\ldots,n)$ is invariant under cyclic permutations of $(1,2,\ldots,n)$.

2. $m(1,2,\ldots,n) = (-1)^n\; m(n,n-1,\ldots,1)$.

3. $m(1,2,\ldots,n)$ is gauge invariant, and satisfies the following identity:

   $$m(1,2,3,\ldots,n)+m(2,1,3,\ldots,n)+m(2,3,1,\ldots,n)+\ldots+m(2,3,\ldots,n-1,1,n) = 0. \tag{3.2}$$

   For future reference we will call this identity the *Dual* Ward Identity (DWI), since from a string point of view it may be thought of as a Ward identity for correlation functions of two dimensional conformal fields. Of course it can also be proved by looking at the Feynman diagram expansion[1].

4. Factorization on the m-particle poles and factorization of the collinear and soft singularities[1,5].

5. Incoherence to leading order in N:

$$\sum_{colors} |A_n|^2 = N^{n-2}(N^2-1) \sum_{perm'} \{|m(1,2,\ldots,n)|^2 + O(N^{-2})\}. \qquad (3.3)$$

It is easy to prove that a similar representation of the amplitude must hold for processes involving gluinos as well as gluons, because the gluino vertices have the same color structure as the gluon ones. Also, we can define gluino sub-amplitudes that satisfy all of the above properties. If we apply the SWI to the amplitude written in the dual basis we obtain a set of identities among sub-amplitudes, and not just among the full amplitudes. Given the simplicity of the gluon sub-amplitudes, one then expects that the gluino sub-amplitudes will be simple as well. In the remaining part of this Section we will show how to use these gluino sub-amplitudes in order to generate an expression similar to Equation (3.1) for amplitudes with a quark-antiquark pair.

To start with, let us recall the procedure for obtaining the gluon sub-amplitudes from the Feynman diagram expansion. First of all we expand the color factor of each Feynman diagram as a sum of traces of λ matrices. This is easily achieved by writing $f_{abc} = -i\, tr[\lambda_a, \lambda_b]\lambda_c/\sqrt{2}$ for one of the vertices, and absorbing the remaining f_{abc}'s by using the identity $i\sqrt{2} f_{abc}\lambda_a = [\lambda_b, \lambda_c]$. In this way, for example, we obtain the following identity:

$$\sum_{x,y,z} f_{12x} f_{x3y} f_{y45} = \left(\frac{1}{i\sqrt{2}}\right)^3 tr[\lambda_1, \lambda_2][\lambda_3, [\lambda_4, \lambda_5]]. \qquad (3.4)$$

This color factor appears, for instance, in a Feynman diagram in which the gluon 1 emits the three gluons 2, 3 and 4, emerging as gluon 5. We will denote a diagram with this structure $D(1,2,3,4,5)$ (see fig. 1). Expanding the commutators of λ matrices we see that this diagram contributes to various different sub-amplitudes, namely those associated with the permutations of indices arising from the expansion of the commutators. To obtain the sub-amplitude corresponding to one given ordering, we just have to identify all of the diagrams that contain a trace of λ's in the order assigned (up to cyclic permutations). In the five gluon case described above, for example, one can easily see that the diagrams contributing to the ordering $(1,2,3,4,5)$ are obtained by taking the five cyclic permutations of the diagram $D(1,2,3,4,5)$ plus diagrams with four-gluon couplings that are necessary for the gauge-invariance of the sub-amplitude :

$$\begin{aligned} m(1,2,3,4,5) &= D^0(1,2,3,4,5) + D^0(2,3,4,5,1) + D^0(3,4,5,1,2) \\ &+ D^0(4,5,1,2,3) + D^0(5,1,2,3,4). \end{aligned} \qquad (3.5)$$

For simplicity we have not shown the diagrams with four-gluon couplings, since they have the same color structure as the simpler diagrams. In taking the sum it is understood that we omit the color factors at the vertices. This is indicated by the superscript 0 added to the D's. The overall color factor $tr(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5) - tr(\lambda_5\lambda_4\lambda_3\lambda_2\lambda_1)$, common to all five diagrams, can be reintroduced at the end of the calculation. The second trace with a minus sign is present because of the property 2 above. In the case of gluinos this construction goes through in exactly the same manner. The calculation of the sub-amplitude will be simpler, though, because some diagrams that were present in the purely gluonic case are absent here. This is the case, for example, for some of the diagrams containing the four-gluon vertex.

Let us now study diagrams containing quarks. For the sake of simplicity we continue with the five-parton example. We use the same notation as before for the diagrams, but now we put a hat on the fermion indices to distinguish them from the gluon ones. We also introduce subscripts $\tilde{g}$ and q to refer to gluino related quantities and quark related quantities, respectively. We start by observing that, up to the color factor, any gluino diagram is identical to the same diagram with quarks replacing the gluinos. From this observation it follows, in particular:

$$\sum_{cyclic} D^0_q(\hat{1},2,3,4,\hat{5}) = \sum_{cyclic} D^0_{\tilde{g}}(\hat{1},2,3,4,\hat{5}) \equiv m_{\tilde{g}}(\hat{1},2,3,4,\hat{5}). \tag{3.6}$$

The sums are over the five cyclic permutations of $(1\ldots5)$. We can now define a quark sub-amplitude and identify it with the gluino one, provided all of the diagrams D_q entering the sum . Equation (3.6) have a common color factor, to be identified with the quark analogue of the Chan-Paton factor.

The color factor for $D_q(\hat{1},2,3,4,\hat{5})$, for example, is given by:

$$\sum_{x,y} \lambda^2_{\hat{1}x}\lambda^3_{xy}\lambda^4_{y\hat{5}} = (\lambda^2\lambda^3\lambda^4)_{\hat{1}\hat{5}}. \tag{3.7}$$

It is easy to check that all of the five diagrams contain this factor. This suggests that the proper color basis to study the amplitude for two quarks and three gluons is given by the product of the three λ matrices corresponding to the colors of the external gluons. A simple analysis of other diagrams shows that it is always possible to expand them in terms of sums of products of the three λ matrices, possibly in different permutations. In fact all of the permutations of $(2,3,4)$ appear.

Some of the diagrams contain additional color factors. For example, $D(\hat{5},\hat{1},2,3,4)$ contains a factor $(\lambda^2\lambda^4\lambda^3)_{\hat{1}\hat{5}}$ in addition to the $(\lambda^2\lambda^3\lambda^4)_{\hat{1}\hat{5}}$ that we have isolated to extract $m_q(\hat{1},2,3,4,\hat{5})$. The presence of this term, nevertheless, should be of no concern, since it is clear from its structure that $D(\hat{5},\hat{1},2,3,4)$ will also have to contribute to the sub-amplitude $m(\hat{1},2,4,3,\hat{5})$.

In a similar way one can analyse all of the sub-amplitudes of the kind $m_q(\hat{1}, i, j, k, \hat{5})$, where (i, j, k) is a permutation of $(2,3,4)$, with the conclusion that

$$m_q(\hat{1}, i, j, k, \hat{5}) = m_{\tilde{g}}(\hat{1}, i, j, k, \hat{5}). \tag{3.8}$$

If we now look at the sub-amplitudes in which the two quarks are not adjacent, we discover that there are no pieces of the diagram left to contribute: they all went into the sub-amplitudes with adjacent quarks. The diagram $D_q(\hat{1}, 2, 3, \hat{5}, 4)$, for example, has already been used to generate $m_q(\hat{1}, 2, 3, 4, \hat{5})$. In the gluino case, vice versa, the same diagram contains a piece proportional to $tr(\lambda_1\lambda_2\lambda_3\lambda_5\lambda_4)$, and this piece will contribute to $m_{\tilde{g}}(\hat{1}, 2, 3, \hat{5}, 4)$.

These considerations can be easily generalized to processes with more than three gluons, and lead to the following expression:

$$A(\bar{q}_1^+, q_2^-, g_3, \ldots, g_n) = \sum_{\{3,\ldots,n\}} (\lambda^3\lambda^4\cdots\lambda^n)_{\hat{2}\hat{1}}\, m(\Lambda_1^+, \Lambda_2^-, g_3, \ldots, g_n). \tag{3.9}$$

$m(\Lambda_1, \Lambda_2, g_3, \ldots, g_n)$ is a sub-amplitude for two gluinos and $(n-2)$ gluons. The expansion of the quark amplitude in terms of this color basis was used by Kunszt in Reference [8]. The advantage of our derivation is the identification of the functions m with gluino sub-amplitudes with the two fermions adjacent. These sub-amplitudes can then be written in a very simple form, as will be shown in the next Section. We will call the color basis in Equation (3.9) the *quark dual basis*.[1]

The quark sub-amplitudes satisfy all of the properties of the gluino sub-amplitudes , except for the DWI. This identity, in fact, requires the introduction of a sub-amplitude with the two fermions not adjacent. Nevertheless, one can still introduce these non-adjacent sub-amplitudes for the quarks as auxiliary functions, that may help simplifying some calculation.

For the amplitude squared, we have an expression very similar to Equation (3.3):

$$\sum_{colors} |A(\bar{q}_1^+, q_2^-, g_3, \ldots, g_n)|^2 = N^{n-3}(N^2-1) \sum_{\{3,\ldots,n\}} \{|m(\Lambda_1^+, \Lambda_2^-, g_3, \ldots, g_n)|^2 + O(N^{-2})\}. \tag{3.10}$$

Notice however the change in the exponent of the leading power of N. We will give the explicit form of the sub-leading terms for $n = 4, 5, 6$ in Appendix C.

4 Five- and Six-Parton Processes

In this Section we will calculate the matrix elements for the scattering of a quark-antiquark pair with three and four gluons. Given the discussion presented in the previous Section, all

[1] Since nowhere in this discussion we actually specified under which representation of $SU(N)$ our quarks transform, the only ingredient for the proof being the commutation relations for the λ matrices, we conclude that the same procedure applies also for fermions in representations other than the fundamental. It is sufficient to replace the λ matrices in Equation (3.9) with the proper matrices in the representation we are interested in.

we need are the sub-amplitudes for two gluinos and three or four gluons. We will obtain these sub-amplitudes by applying the supersymmetry trick to the five- and six-gluon sub-amplitudes, calculated elsewhere[1,2]. In order to be self-contained, we will also recall the results for the gluon amplitudes.

In the five gluon case there is only one independent helicity amplitude, with two negative- and three positive-helicity gluons. The amplitude with three negative- and two positive-helicity gluons is obtained from this one by replacing $\langle . . \rangle$ with $[. .]$. All of the relevant sub-amplitudes can be obtained from the following one by proper permutations:

$$m_{3+2-}(g_1, g_2, g_3, g_4, g_5) \;=\; ig^3 \frac{\langle IJ\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}. \tag{4.1}$$

Here I and J are the momenta of the negative helicity gluons, and the ordering in the denominator is determined by the order of the momenta in the sub-amplitude. If we now use equations (2.8) and (3.9) we get the following result for the full quark amplitude:

$$A(\bar{q}_1^+, q_2^-, g_3^-, g_4^+, g_5^+) = ig^3\langle 23\rangle^3\langle 13\rangle \sum_{\{3,4,5\}} (\lambda^3\lambda^4\lambda^5)_{\hat{2}\hat{1}} \frac{1}{\langle 12\rangle\langle 23\rangle\cdots\langle 51\rangle}, \tag{4.2}$$

where we defined q^- to be a negative-helicity quark and $\bar{q}^+$ is its anti-quark. A similar expression holds for the two quark-two gluon case. It is straightforward to check that Equation (4.2) agrees with the result already known for the amplitude squared[13], see Appendix C.

The six-parton processes are more complex[7,8,9]. Two independent helicity amplitudes are needed: A_{2-4+} and A_{3-3+}. The first one is a trivial generalization of the five-parton amplitude, and the full amplitude for two quarks and four gluons is given by:

$$A(\bar{q}_1^+, q_2^-, g_3^-, g_4^+, \ldots, g_6^+) = ig^4\langle 23\rangle^3\langle 13\rangle \sum_{\{3,4,5,6\}} (\lambda^3\lambda^4\lambda^5\lambda^6)_{\hat{2}\hat{1}} \frac{1}{\langle 12\rangle\langle 23\rangle\cdots\langle 61\rangle}. \tag{4.3}$$

The six-gluon helicity amplitude A_{3-3+} is described by three distinct sub-amplitudes, characterised by three inequivalent helicity orderings: $(+++---)$, $(++-+--)$ and $(+-+-+-)$. All of these sub-amplitudes can be written in the following form[1]:

$$\begin{aligned} m_{3+3-}(g_1, g_2, g_3, g_4, g_5, g_6) \quad &= \quad ig^4 \left[\frac{\alpha^2}{t_{123}S_{12}S_{23}S_{45}S_{56}} + \frac{\beta^2}{t_{234}S_{23}S_{34}S_{56}S_{61}} \right. \\ &\quad + \left. \frac{\gamma^2}{t_{345}S_{34}S_{45}S_{61}S_{12}} + \frac{t_{123}\beta\gamma + t_{234}\gamma\alpha + t_{345}\alpha\beta}{S_{12}S_{23}S_{34}S_{45}S_{56}S_{61}} \right], \end{aligned} \tag{4.4}$$

where $t_{ijk} \equiv (p_i + p_j + p_k)^2 \;=\; S_{ij} + S_{jk} + S_{ki}$. The coefficients α, β and γ for the three distinct orderings of helicities are given in Table 1.

To obtain the fermionic sub-amplitudes we now need the proper SWI. It is convenient to calculate the expectation value of the following commutator: $[Q(k), g_1^+ \Lambda_2^- g_3^- g_4^+ g_5^- g_6^-]$. By expanding the commutator and using equations (2.1) and (2.2) we obtain:

$$-\Gamma^+(p_1,k)A(\Lambda_1^+,\Lambda_2^-,g_3^-,g_4^+,g_5^-,g_6^-)+\Gamma^+(p_2,k)A(g_1^+,g_2^-,g_3^-,g_4^+,g_5^-,g_6^-)$$

$$+\Gamma^+(p_3,k)A(g_1^+,\Lambda_2^-,\Lambda_3^-,g_4^+,g_5^-,g_6^-)+\Gamma^+(p_4,k)A(g_1^+,\Lambda_2^-,g_3^-,\Lambda_4^+,g_5^-,g_6^-)=0. \qquad (4.5)$$

Helicity conservation has been used to cancel the two amplitudes with two negative-helicity fermions, and the Grassmannian nature of $\Gamma^\pm$ was used when moving it through Λ_2[10]. If we now choose $k = p_4$ we are left with a relation between a purely gluonic amplitude and two fermionic ones. This means that the gluonic amplitudes alone are not sufficient to determine the fermionic ones. If we had to start this calculation from scratch, it would be better to calculate the fermionic amplitudes first, and then from these obtain the gluonic ones. This is the technique that was used, for example, in [6,7,8,9]. Since we already have in a simple form the sub-amplitudes for the gluons, we would like to use these without having to calculate any new Feynman diagrams. We will show how to disentangle Equation(4.5) in Appendix B.

The distinct helicity sub-amplitudes that we need are the following:

(I) $m(\bar{q}_1^+, q_2^-, g_3^-, g_4^-, g_5^+, g_6^+)$;

(II) $m(\bar{q}_1^+, q_2^-, g_3^+, g_4^+, g_5^-, g_6^-)$;

(III) $m(\bar{q}_1^+, q_2^-, g_3^-, g_4^+, g_5^+, g_6^-)$;

(IV) $m(\bar{q}_1^+, q_2^-, g_3^+, g_4^-, g_5^-, g_6^+)$;

(V) $m(\bar{q}_1^+, q_2^-, g_3^-, g_4^+, g_5^-, g_6^+)$;

(VI) $m(\bar{q}_1^+, q_2^-, g_3^+, g_4^-, g_5^+, g_6^-)$.

The amplitudes with positive helicity quarks can be obtained by complex conjugation and reordering of the sub-amplitudes.

When the SWI's are applied to the sub-amplitudes, the ordering of the helicities is not changed. This implies that only (II), (III), (IV) and (V) can be related through a SWI. However we also know that the fermionic sub-amplitudes, as the gluonic ones, obey a further identity, namely the Dual Ward Identity, equation (3.2). This identity reshuffles the helicities, and allows us to obtain relations among sub-amplitudes with different helicity orderings, see Appendix B.

The general form of the sub-amplitudes is dictated by duality[14], which gives the following pole structure:

$$m(\bar{q}_1, q_2, g_3, g_4, g_5, g_6) \quad = \quad -ig^4 \left[\frac{P_1}{t_{123}S_{12}S_{23}S_{45}S_{56}} + \frac{P_2}{t_{234}S_{23}S_{34}S_{56}S_{61}} \right.$$

$$+ \quad \frac{P_3}{t_{345}S_{34}S_{45}S_{61}S_{12}} + \frac{P_s}{S_{12}S_{23}S_{34}S_{45}S_{56}S_{61}} \Big] . \tag{4.6}$$

We group the expressions for the functions P in two tables, Table 2 for the P_i's, $i = 1,2,3$, and Table 3 for P_s. The resulting amplitude squared summed over colors, see Appendix C, agrees numerically to machine precision with previously published results[7,8]. This concludes the analysis of the $(\bar{q}qgggg)$ process.

5 Other Examples

In this Section we briefly analyse further processes. Most of them will only become of direct interest at the energies available in future accelerators (SSC, LHC), and we present them mainly to show the versatility of the techniques that have been introduced in this paper.

To start with we give the set of SWI's which are necessary to describe the two massless gluino-four gluon scattering process. The sub-amplitudes with two adjacent gluinos were given in the previous section, when discussing the quark process. The remaining sub-amplitudes with non-adjacent gluinos can be obtained from the following SWI's:

$$[41]m(\Lambda_1^+, g_2^+, \Lambda_3^-, g_4^+, g_5^-, g_6^-) = [43]m(g_1^+, g_2^+, g_3^-, g_4^+, g_5^-, g_6^-) - [42]m(g_1^+, \Lambda_2^+, \Lambda_3^-, g_4^+, g_5^-, g_6^-), \tag{5.1}$$

$$[61]m(\Lambda_1^+, g_2^-, \Lambda_3^-, g_4^-, g_5^-, g_6^+) = [63]m(g_1^+, g_2^+, g_3^-, g_4^-, g_5^-, g_6^-) - [62]m(g_1^+, \Lambda_2^+, \Lambda_3^-, g_4^-, g_5^-, g_6^+), \tag{5.2}$$

$$[21]m(\Lambda_1^+, g_2^+, g_3^+, \Lambda_4^-, g_5^-, g_6^-) = [24]m(g_1^+, g_2^+, g_3^+, g_4^-, g_5^-, g_6^-) - [23]m(g_1^+, g_2^+, \Lambda_3^+, \Lambda_4^-, g_5^-, g_6^-), \tag{5.3}$$

$$[21]m(\Lambda_1^+, g_2^+, g_3^-, \Lambda_4^-, g_5^+, g_6^-) = [24]m(g_1^+, g_2^+, g_3^-, g_4^-, g_5^+, g_6^-) + [25]m(g_1^+, g_2^+, g_3^-, \Lambda_4^-, \Lambda_5^+, g_6^-), \tag{5.4}$$

$$[31]m(\Lambda_1^+, g_2^-, g_3^+, \Lambda_4^-, g_5^+, g_6^-) = [34]m(g_1^+, g_2^-, g_3^+, g_4^-, g_5^+, g_6^-) + [35]m(g_1^+, g_2^-, g_3^+, \Lambda_4^-, \Lambda_5^+, g_6^-), \tag{5.5}$$

$$[61]m(\Lambda_1^+, g_2^+, g_3^-, \Lambda_4^-, g_5^-, g_6^+) = [64]m(g_1^+, g_2^+, g_3^-, g_4^-, g_5^-, g_6^-) - [62]m(g_1^+, \Lambda_2^+, g_3^-, \Lambda_4^-, g_5^-, g_6^+). \tag{5.6}$$

The new sub-amplitudes generated by each SWI are put on the lefthand side. Any sub-amplitude needed in these equations that has not been explicitly calculated before its appearance is trivially obtained from already known expressions by complex conjugation or reordering of the indices. The same holds for other sub-amplitudes not appearing in this list, as for example $m(\Lambda_1^+, g_2^-, g_3^-, \Lambda_4^-, g_5^-, g_6^+)$.

Next we derive exact expressions for some sets of helicity amplitudes. One of these expressions was already given in Equation (2.8), namely the form of the helicity amplitude $A(\Lambda_1^+, \Lambda_2^-, g_3^-, g_4^+, \ldots, g_n^+)$, describing the most helicity-violating scattering[2] of two gluinos and $(n-2)$ gluons:

$$A_{\tilde{g}}(\Lambda_1^+, \Lambda_2^-, g_3^-, g_4^+, \ldots, g_n^+) = ig^{n-2}\langle 23\rangle^3\langle 13\rangle \sum_{perm'} tr(\lambda_1\lambda_2\cdots\lambda_n)\frac{1}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}. \tag{5.7}$$

If we now use Equation (3.9) we can directly obtain the amplitude for an equivalent process, involving quarks instead of gluinos:

$$A_q(\bar{q}_1^+, q_2^-, g_3^-, g_4^+, \ldots, g_n^+) = ig^{n-2}\langle 23\rangle^3\langle 13\rangle \sum_{\{3,\ldots,n\}} (\lambda_3\lambda_4\cdots\lambda_n)_{\hat{2}\hat{1}} \frac{1}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}. \tag{5.8}$$

This formula was guessed in Reference [5] by studying the behaviour of gluon amplitudes in the limit of collinear gluon emission.

Let us now take the amplitude $A(\Lambda_1^+, \Lambda_2^-, \Lambda_3^+, g_4^-, g_5^+, \ldots, g_n^+)$. By commuting with the supersymmetry operator and properly choosing the reference momentum k we obtain the following SWI:

$$A_{\tilde{g}}(\Lambda_1^+, \Lambda_2^+, \Lambda_3^-, \Lambda_4^-, g_5^+, \ldots, g_n^+) = \frac{\langle 12\rangle}{\langle 24\rangle} A_{\tilde{g}}(g_1^+, \Lambda_2^+, \Lambda_3^-, g_4^-, g_5^+, \ldots, g_n^+). \tag{5.9}$$

By using Equation (5.7) we get:

$$A_{\tilde{g}}(\Lambda_1^+, \Lambda_2^+, \Lambda_3^-, \Lambda_4^-, g_5^+, \ldots, g_n^+) = ig^{n-2}\langle 12\rangle\langle 34\rangle^3 \sum_{perm'} tr(\lambda_1\lambda_2\cdots\lambda_n)\frac{1}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}. \tag{5.10}$$

For $n=6$ the missing helicity amplitudes can be easily obtained by use of the SWI's and the two gluino-four gluon sub-amplitudes given at the beginning of this Section.

It is not possible to directly relate the sub-amplitudes for a four-gluino process to sub-amplitudes for a four-quark process. This is clear for the scattering of two pairs of quarks of different flavour: some diagrams that are present in the gluino case are absent for the quarks because it is not possible to contract two different-quark lines. Even if the two quark pairs have the same flavour, though, the gluino sub-amplitudes are different from the quark ones. The reason for this being that diagrams containing a contraction between adjacent quarks have a different color factor from diagrams with a contraction between non-adjacent quarks. As an example of this fact, take for instance the gluino sub-amplitude $m_{\tilde{g}}(\Lambda_1^+, \Lambda_2^-, \Lambda_3^+, \Lambda_4^-, g_5, g_6)$, where for our purposes now the gluon helicities are irrelevant. This sub-amplitude is generated by the sum over

[2] We refer to the non-zero helicity-violating processes. The amplitudes $(+ + \ldots + +)$ and $(- + + \ldots +)$ vanish identically.

cyclic permutations of the diagram $D_{\tilde{g}}(\hat{1},\hat{2},\hat{3},\hat{4},5,6)$ plus diagrams with four-gluon couplings. Let us concentrate on two of these diagrams, namely $D_{\tilde{g}}(\hat{1},\hat{2},\hat{3},\hat{4},5,6)$ and $D_{\tilde{g}}(\hat{4},5,6,\hat{1},\hat{2},\hat{3})$, and let us label them as $D^I_{\tilde{g}}$ and $D^{II}_{\tilde{g}}$. $D^I_{\tilde{g}}$ has fermion 1 contracted with fermion 2, and fermion 3 contracted with fermion 4. $D^{II}_{\tilde{g}}$ has fermion 1 contracted with fermion 4 and fermion 2 contracted with fermion 3. If the two quark pairs $(q_1,\bar{q}_2)$ and $(q_3,\bar{q}_4)$ are of different flavour, it clearly follows that $D^{II}_q = 0$. If the two quark pairs are identical, then $(D^I_q)^0 = (D^I_{\tilde{g}})^0$ and $(D^{II}_q)^0 = (D^{II}_{\tilde{g}})^0$. However, the color factors for the two diagrams are given by the following expressions:

$$D^I_q \longrightarrow \delta_{\hat{2}\hat{3}}[\lambda^5,\lambda^6]_{\hat{4}\hat{1}} \;-\; \frac{1}{N}\delta_{\hat{1}\hat{2}}[\lambda^5,\lambda^6]_{\hat{4}\hat{3}}, \tag{5.11}$$

$$D^{II}_q \longrightarrow \delta_{\hat{1}\hat{2}}(\lambda^5\lambda^6)_{\hat{4}\hat{3}} \;-\; \frac{1}{N}\delta_{\hat{2}\hat{3}}(\lambda^5\lambda^6)_{\hat{4}\hat{1}}. \tag{5.12}$$

Since the two diagrams do not have a common color factor, it is not clear how to define a quark sub-amplitude for this process. We do not think that this is a serious drawback of the technique. The color structure of a four quark-two gluon process is simple and the number of diagrams is relatively small. A direct calculation of the matrix element is then possible, and was performed by Gunion and Kunszt[15] and by Z. Xu *et al.*[11], who found a very compact analytic expression.

To conclude we give the most helicity-violating amplitude for the scattering of gluons and a pair of massless scalar-quarks, obtained from the SWI and the supersymmetry transformations of a chiral superfield[10]:

$$A(\bar{\phi}^+_1,\phi^-_2,g^-_3,g^+_4,\ldots,g^+_n) = ig^{n-2}\langle 23\rangle^2\langle 13\rangle^2 \sum_{\{3,\ldots,n\}} (\lambda_3\lambda_4\cdots\lambda_n)_{\hat{2}\hat{1}}\, \frac{1}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}. \tag{5.13}$$

$\phi^\pm$ are the supersymmetry partners of the two helicity states of the quark. The two combinations $\phi^+ \pm \phi^-$ transform respectively as a scalar and a pseudoscalar under the Lorentz group. For $n = 4,5$ these are the only independent non-vanishing helicity amplitudes for this process.

6 Conclusions

In this paper we have generalized the *dual-expansion* technique to processes involving particles other than gluons. The extension to gluinos is straightforward, and with minor modifications it is possible to treat the quark-antiquark multi-gluon amplitudes. The use of supersymmetry Ward identities allows us to relate sub-amplitudes with particles of different spin, thus reducing substantially the amount of calculations.

We have explicitly recalculated the full matrix element for the $(q\bar{q}gggg)$ process in this formalism, and we have found a significant simplification compared to the result already known in the literature. We have also given in implicit form the amplitude for two gluino-four gluon scattering and explicit formulae for the mostly helicity-violating amplitudes for processes with two quarks and n gluons, two gluinos and n gluons, four gluinos and n gluons and two scalar-quarks and n gluons. We believe all these examples prove that the dual expansion, the supersymmetry Ward identities and the improved Calkul method merge into an extremely powerful technique to efficiently calculate tree amplitudes in massless QCD.

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A Appendix

In this Appendix we introduce our notation. Here and in the rest of the paper we adopt the Xu *et al. improved* version[11] of the CALKUL parametrization for the helicities[12]. We define the following symbols for chiral spinors and *spinor products*[11]:

$$|i\pm\rangle = u^{(\pm)}(p_i) = \frac{1}{2}(1\pm\gamma_5)u^{(\pm)}(p_i)\ , \qquad \langle i\pm| = \bar{u}^{(\pm)}(p_i) = \bar{u}^{(\pm)}(p_i)(1\mp\gamma_5)\frac{1}{2} \tag{A.1}$$

$$\langle ij\rangle = \langle i-|j+\rangle\ , \qquad [ij] = \langle i+|j-\rangle = sign(p_i^0 p_j^0)\langle ji\rangle^*. \tag{A.2}$$

We recall here some of the properties satisfied by these symbols[11]:

$$[pp] = \langle pp\rangle = \langle p+|q+\rangle = \langle p-|q-\rangle = 0, \tag{A.3}$$

$$\langle pq\rangle = -\langle qp\rangle, \qquad [pq] = -[qp], \tag{A.4}$$

$$|p\pm\rangle\langle p\pm| = \frac{1}{2}(1\pm\gamma_5)p\cdot\gamma, \qquad \langle p|k|q\rangle \equiv \langle p+|k\cdot\gamma|q+\rangle = [pk]\langle kq\rangle, \tag{A.5}$$

$$\langle p+|\gamma^\mu|q+\rangle\langle k-|\gamma^\mu|l-\rangle = 2[pl]\langle kq\rangle, \qquad \langle pq\rangle\langle kl\rangle = \langle pl\rangle\langle kq\rangle + \langle pk\rangle\langle ql\rangle. \tag{A.6}$$

B Appendix

In this Appendix we collect the Supersymmetry Ward Identities necessary to calculate the sub-amplitudes $m(\bar{q}qgggg)$. We need the sub-amplitudes for seven independent helicity configurations, as explained in Section 4:

(I) $m(\Lambda_1^+,\Lambda_2^-,g_3^-,g_4^-,g_5^+,g_6^+)$;

(II) $m(\Lambda_1^+,\Lambda_2^-,g_3^+,g_4^+,g_5^-,g_6^-)$;

(III) $m(\Lambda_1^+,\Lambda_2^-,g_3^-,g_4^+,g_5^+,g_6^-)$;

(IV) $m(\Lambda_1^+,\Lambda_2^-,g_3^+,g_4^-,g_5^-,g_6^+)$;

(V) $m(\Lambda_1^+,\Lambda_2^-,g_3^-,g_4^+,g_5^-,g_6^+)$;

(VI) $m(\Lambda_1^+,\Lambda_2^-,g_3^+,g_4^-,g_5^+,g_6^-)$;

(VII) $m(\Lambda_1^+,\Lambda_2^-,g_3^-,g_4^+,g_5^+,g_6^+)$.

The sub-amplitude (VII) was described in Section 4, and we will not consider it again here. As we pointed out in Section 4 not all of the fermionic sub-amplitudes can be expressed directly in terms of gluonic ones. By playing with the SWI and the DWI it is easy to convince yourself that only two sub-amplitudes are needed to obtain all of the others. Suppose in fact we have (I) and (II). Then we can get (III) from (II) by using the following form of the SWI:

$$\langle 51\rangle m(g_1^+,g_2^-,g_3^+,g_4^+,g_5^-,g_6^-)+\langle 52\rangle m(\Lambda_1^+,\Lambda_2^-,g_3^+,g_4^+,g_5^-,g_6^-)+\langle 56\rangle m(\Lambda_1^+,\Lambda_6^-,g_5^-,g_4^+,g_3^+,g_2^-)=0. \tag{B.1}$$

Given (III) we can easily get (IV) by replacing $[IJ]$ with $\langle IJ\rangle$ and changing the overall sign. With the help of (IV) we obtain the sub-amplitude (V) by solving the following system:

$$\begin{cases} [41]m(\Lambda_1^+,\Lambda_2^-,g_3^-,g_4^+,g_5^-,g_6^+)-[46]m(g_1^+,\Lambda_2^-,g_3^-,g_4^+,g_5^-,\Lambda_6^+)=[42]m(g_1^+,g_2^-,g_3^-,g_4^+,g_5^-,g_6^+) \\ \langle 32\rangle m(g_1^+,\Lambda_2^-,g_3^-,g_4^+,g_5^-,\Lambda_6^+)+\langle 35\rangle m(\Lambda_6^+,\Lambda_5^-,g_4^+,g_3^-,g_2^-,g_1^+)=\langle 36\rangle m(g_1^+,g_2^-,g_3^-,g_4^+,g_5^-,g_6^+) \end{cases} \tag{B.2}$$

To calculate the sub-amplitude (VI) we need a DWI:

$$m(\Lambda_1^-,\Lambda_2^-,g_3^+,g_4^-,g_5^+,g_6^-)+m(\Lambda_1^+,\Lambda_2^-,g_3^+,g_4^-,g_6^-,g_5^+)+m(\Lambda_1^+,g_5^+,\Lambda_2^-,g_3^+,g_4^-,g_6^-)$$

$$+m(\Lambda_1^+,\Lambda_2^-,g_5^+,g_3^+,g_4^-,g_6^-)+m(\Lambda_1^+,\Lambda_2^-,g_3^+,g_5^+,g_4^-,g_6^-)=0. \tag{B.3}$$

The sub-amplitude with non-adjacent fermions needed to solve equation (B.3) follows from this

SWI:

$$[31]m(\Lambda_1^+, g_5^+, \Lambda_2^-, g_3^+, g_4^-, g_6^-) = [35]m(\Lambda_5^+, \Lambda_2^-, g_3^+, g_4^-, g_6^-, g_1^-) - [32]m(g_1^+, g_5^+, g_2^-, g_3^-, g_4^-, g_6^-). \tag{B.4}$$

This completes the set of sub-amplitudes necessary to calculate the two quark-four gluon amplitude, provided we supply (I) and (II).

To obtain (I) and (II), one may have a priori to calculate the relevant set of Feynman diagrams, which is not a very difficult task. However we can avoid even this nuisance if we study the behaviour of (I) and (II) in the proximity of the three particle poles and in the limit in which two of the partons become collinear.

Let us first consider (II), and let us write the following two SWI that involve (II):

$$\langle 62\rangle m(\Lambda_1^+, \Lambda_2^-, g_3^-, g_4^+, g_5^-, g_6^-) + \langle 65\rangle m(\Lambda_1^+, g_2^-, g_3^+, g_4^+, \Lambda_5^-, g_6^-) + \langle 61\rangle m(g_1^+, g_2^-, g_3^+, g_4^+, g_5^-, g_6^-) = 0 \tag{B.5}$$

$$[31]m(\Lambda_1^+, \Lambda_2^-, g_3^+, g_4^+, g_5^-, g_6^-) - [34]m(g_1^+, \Lambda_2^-, g_3^+, \Lambda_4^+, g_5^-, g_6^-) - [32]m(g_1^+, g_2^-, g_3^+, g_4^+, g_5^-, g_6^-) = 0 \tag{B.6}$$

We can use these two identities and duality to derive (II). In fact, from duality[14] we know that the fermionic sub-amplitudes have the following pole structure:

$$\begin{aligned} m(1,2,3,4,5,6) \quad &= \quad -ig^4\left[\frac{P_1}{t_{123}S_{12}S_{23}S_{45}S_{56}} + \frac{P_2}{t_{234}S_{23}S_{34}S_{56}S_{61}}\right. \\ &+ \quad \left.\frac{P_3}{t_{345}S_{34}S_{45}S_{61}S_{12}} + \frac{P_s}{S_{12}S_{23}S_{34}S_{45}S_{56}S_{61}}\right]. \end{aligned} \tag{B.7}$$

The functions P_i, $(i = 1,2,3)$, exhibit the factorization of the sub-amplitude on the three particle poles. Since any sub-amplitude $m(\Lambda^+, \Lambda^-, g^+, g^+)$ vanishes identically, it immediately follows that $P_2 = 0$ for (II). For the same reason $P_3 = 0$ for the second fermionic sub-amplitude in equation (B.5) and $P_1 = 0$ for the second fermionic sub-amplitude in equation (B.6). In this way we can read out P_1 and P_3 for (II) directly from Table 1 and equations (B.5), (B.6). Given P_1 and P_3, the remaining function P_s can be obtained by imposing the proper behaviour of the collinear singularities[16].

The same steps can be followed for (I). The explicit expressions for the sub-amplitudes (I),...,(VII) are contained in Section 4. The resulting amplitude squared has been checked numerically against the available expressions previously obtained in reference [7,8]. The agreement is to machine precision.

C Appendix

In this Appendix we give the structure of the amplitude squared for the processes $(\bar{q}qgg)$, $(\bar{q}qggg)$ and $(\bar{q}qgggg)$. To keep the following formulae as simple as possible, we introduce the following notation for the quark sub-amplitudes :

$$m(\bar{q}_1, q_2, g_I, g_J, \ldots, g_L) = (I, J, \ldots, L), \tag{C.1}$$

where $(I, J, \ldots, L)$ is an arbitrary permutation of $(3, 4, \ldots, 6)$. From the expansion of the amplitude in the quark dual basis,

$$A(\bar{q}_1, q_2, g_3, \ldots, g_n) = \sum_{\{3,\ldots,n\}} (\lambda^3\lambda^4 \ldots \lambda^n)_{\bar{2}\bar{1}}\ (3, 4, \ldots, n), \tag{C.2}$$

we obtain the following expression:

$$\sum_{colors} |A(\bar{q}_1, q_2, g_3, \ldots, g_n)|^2 = \frac{(N^2-1)}{N^{n-3}} \sum_{j=0}^{n-3} N^{2j} \sum_{\{3,\ldots,n\}} H_j(3, 4, \ldots, n). \tag{C.3}$$

For $n = 4, 5, 6$ the functions H_j are given by:

- $n = 4$

$$H_1(3,4) = |(3,4)|^2 \tag{C.4}$$

$$H_0(3,4) = -(3,4)^*[(3,4) + (4,3)] \tag{C.5}$$

- $n = 5$

$$H_2(3,4,5) = |(3,4,5)|^2 \tag{C.6}$$

$$H_1(3,4,5) = -(3,4,5)^*[2\ (3,4,5) + (3,5,4) + (4,3,5) + (5,4,3)] \tag{C.7}$$

$$H_0(3,4,5) = (3,4,5)^* \sum_{\{I,J,K\}} (I,J,K) \tag{C.8}$$

- $n = 6$

$$H_3(3,4,5,6) = |(3,4,5,6)|^2, \tag{C.9}$$

$$\begin{aligned} H_2(3,4,5,6) = (3,4,5,6)^*[&-3\ (3,4,5,6) - (3,4,6,5) - (3,5,4,6) \\ &-(4,3,5,6) + (3,6,5,4) + (5,4,3,6) + (5,6,3,4) \\ &+(5,6,4,3) + (6,4,5,3) + (6,5,3,4)], \end{aligned} \tag{C.10}$$

$$H_1(3,4,5,6) = (3,4,5,6)^*[M(3,4,5,6) - M(6,5,4,3)] \tag{C.11}$$

$$\begin{aligned} M(3,4,5,6) &= 3\,(3,4,5,6) + 2\,(3,4,6,5) + 2\,(3,5,4,6) + 2\,(4,3,5,6) + (3,5,6,4) \\ &\quad +(3,6,4,5) + (4,3,6,5) + (4,5,3,6) + (5,3,4,6), \end{aligned} \tag{C.12}$$

$$H_0(3,4,5,6) = -(3,4,5,6)^* \sum_{\{I,J,K,L\}} (I,J,K,L). \tag{C.13}$$

The formulae for $n = 4,5$ can be used to compare our results with the expressions already known. In doing this it is useful to apply the DWI to the functions H_j and use the gluino sub-amplitudes with non-adjacent fermions as auxiliary functions.

References

[1] M. Mangano, S. Parke and Z. Xu, FERMILAB–Conf–87/78-T, March 1987; "Les Rencontres de Physique de la Vallee d'Aoste", La Thuile, Italy, March 1-7,1987; FERMILAB–Pub–87/52-T, June 1987, to be published in Nucl.Phys..

[2] F. A. Berends and W. Giele, Leiden University Preprint, June 1987.

[3] J. Paton and Chan Hong-Mo, Nucl. Phys. B10 (1969) 519.

[4] E. Eichten, I. Hinchliffe, K. Lane and C. Quigg, Rev. Mod. Phys. 56 (1984), 579.

[5] M. Mangano and S. Parke, FERMILAB–Conf–87/121-T, July 1987, to appear in the Proceedings of the 'International Europhysics Conference on High Energy Physics', Uppsala, Sweden, June 25-July 1, 1987.

[6] S. Parke and T. Taylor, Phys. Lett. 157B (1985), 81.

[7] S. Parke and T. Taylor, Nucl. Phys. B269 (1986), 410;
S. Parke and T. Taylor, Phys. Rev. D35 (1987), 313.

[8] Z. Kunszt, Nucl. Phys. B271 (1986), 333.

[9] J. Gunion and J. Kalinowski, Phys. Rev. D34 (1986), 2119.

[10] M. T. Grisaru, H. N. Pendleton and P. van Nieuwenhuizen, Phys. Rev. D15 (1977) 996;
M. T. Grisaru and H. N. Pendleton, Nucl. Phys. B124 (1977), 81

[11] Z. Xu, Da-Hua Zhang and L. Chang, Tsinghua University Preprints, Beijing, The People's Republic of China, TUTP–84/4, TUTP–84/5, TUTP–84/6 and Nucl. Phys. B291 (1987) 392. J. F. Gunion and Z. Kunszt, Phys. Lett. 161B (1985), 333; R. Kleiss and W. J. Stirling, Nucl. Phys. B262 (1985) 167.

[12] P. De Causmaeker, R. Gastmans, W. Troost and T.T. Wu, Nucl. Phys. B206 (1982), 53; F. A. Berends, R. Kleiss, P. De Causmaeker, R. Gastmans, W. Troost and T.T. Wu, Nucl. Phys. B206 (1982), 61; *ibidem* B264 (1986), 265 and references there in; D. Danckaert, P. De Causmaeker, R. Gastmans, W. Troost and T.T. Wu, Phys. Lett. 114B (1982), 203.

[13] T. Gottschalk and D. Sivers, Phys. Rev. D21 (1980), 102; F. A. Berends, R. Kleiss, P. De Causmaeker, R. Gastmans and T.T. Wu, Phys. Lett. 103B (1981), 124.

[14] See for example M. Jacob and S. Mandelstam's lectures contained in 'Lectures on Elementary Particles and Quantum Field Theory', Brandeis 1970, S. Deser, M. Grisaru and H. Pendleton eds., MIT Press and reference contained in these lectures.

[15] J. Gunion and Z. Kunszt, Phys. Lett. 159B (1985) 167.

[16] G. Altarelli and G. Parisi, Nucl. Phys. B126 (1977), 298.

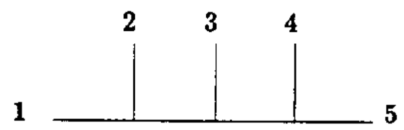

Figure 1: The generic five parton Feynman diagram.

Table 1: Coefficients for the $m_{3+3-}(g_1, g_2, g_3, g_4, g_5, g_6)$ sub-amplitudes with $\langle I|K|J\rangle \equiv \langle I+|K\cdot\gamma|J+\rangle$

	$1^+2^+3^+4^-5^-6^-$ $X = p_1 + p_2 + p_3$	$1^+2^+3^-4^+5^-6^-$ $Y = p_1 + p_2 + p_4$	$1^+2^-3^+4^-5^+6^-$ $Z = p_1 + p_3 + p5$
α	0	$-[12]\langle 56\rangle\langle 4\|Y\|3\rangle$	$[13]\langle 46\rangle\langle 5\|Z\|2\rangle$
β	$[23]\langle 56\rangle\langle 1\|X\|4\rangle$	$[24]\langle 56\rangle\langle 1\|Y\|3\rangle$	$[51]\langle 24\rangle\langle 3\|Z\|6\rangle$
γ	$[12]\langle 45\rangle\langle 3\|X\|6\rangle$	$[12]\langle 35\rangle\langle 4\|Y\|6\rangle$	$[35]\langle 62\rangle\langle 1\|Z\|4\rangle$

Table 2: The numerator functions P_i for $m(\bar{q}_1^+, q_2^-, g_3, g_4, g_5, g_6)$. The left column contains the helicity orderings of the gluons and $\langle I|K|J\rangle \equiv \langle I+|K\cdot\gamma|J+\rangle$.

(g_3, g_4, g_5, g_6)	P_1 $U = p_1 + p_2 + p_3$	P_2 $V = p_2 + p_3 + p_4$	P_3 $W = p_3 + p_4 + p_5$
$(-,-,+,+)_{(I)}$	$[56]^2\langle 13\rangle\langle 23\rangle\langle 1\|U\|4\rangle^2$	0	$-[16][26]\langle 34\rangle^2\langle 5\|W\|2\rangle^2$
$(+,+,-,-)_{(II)}$	$-[13][23]\langle 56\rangle^2\langle 4\|U\|2\rangle^2$	0	$[34]^2\langle 16\rangle\langle 26\rangle\langle 1\|W\|5\rangle^2$
$(-,+,+,-)_{(III)}$	$[45]^2\langle 13\rangle\langle 23\rangle\langle 1\|U\|6\rangle^2$	$[15]\langle 23\rangle\langle 5\|V\|3\rangle\langle 4\|V\|6\rangle^2$	$[45]^2\langle 16\rangle\langle 26\rangle\langle 1\|W\|3\rangle^2$
$(+,-,-,+)_{(IV)}$	$-\langle 45\rangle^2[13][23]\langle 6\|U\|2\rangle^2$	$[16]\langle 24\rangle\langle 6\|V\|4\rangle\langle 3\|V\|5\rangle^2$	$-\langle 45\rangle^2[16][26]\langle 3\|W\|2\rangle^2$
$(-,+,-,+,)_{(V)}$	$[46]^2\langle 13\rangle\langle 23\rangle\langle 1\|U\|5\rangle^2$	$[16]\langle 23\rangle\langle 6\|V\|3\rangle\langle 4\|V\|5\rangle^2$	$-[16][26]\langle 35\rangle^2\langle 4\|W\|2\rangle^2$
$(+,-,+,-,)_{(VI)}$	$-[13][23]\langle 46\rangle^2\langle 5\|U\|2\rangle^2$	$[15]\langle 24\rangle\langle 5\|V\|4\rangle\langle 3\|V\|6\rangle^2$	$[35]^2\langle 16\rangle\langle 26\rangle\langle 1\|W\|4\rangle^2$

Table 3: The numerator functions P_s for $m(\bar{q}_1^+, q_2^-, g_3, g_4, g_5, g_6)$ with the same notation as Table 2.

(g_3, g_4, g_5, g_6)	P_s
$(-,-,+,+)_{(I)}$	$\langle 23\rangle\langle 34\rangle[56][61]\left(\langle 1\vert U\vert 4\rangle\langle 2\vert V\vert 1\rangle\langle 5\vert W\vert 2\rangle - S_{56}[23]\langle 34\rangle\langle 5\vert W\vert 2\rangle - S_{34}[56]\langle 61\rangle\langle 1\vert U\vert 4\rangle\right)$
$(+,+,-,-)_{(II)}$	$[23][34]\langle 56\rangle\langle 61\rangle\langle 4\vert U\vert 2\rangle\langle 1\vert V\vert 2\rangle\langle 1\vert W\vert 5\rangle$
$(-,+,+,-)_{(III)}$	$-t_{123}[15][45]\langle 13\rangle\langle 26\rangle\langle 4\vert V\vert 6\rangle\langle 1\vert W\vert 3\rangle - t_{234}[45]^2\langle 13\rangle\langle 26\rangle\langle 1\vert U\vert 6\rangle\langle 1\vert W\vert 3\rangle$ $+t_{345}[15][45]\langle 13\rangle\langle 23\rangle\langle 1\vert U\vert 6\rangle\langle 4\vert V\vert 6\rangle + [45][56]\langle 12\rangle\langle 36\rangle\langle 1\vert U\vert 6\rangle\langle 4\vert V\vert 6\rangle\langle 1\vert W\vert 3\rangle$
$(+,-,-,+)_{(IV)}$	$t_{123}[16][26]\langle 24\rangle\langle 45\rangle\langle 3\vert V\vert 5\rangle\langle 3\vert W\vert 2\rangle + t_{234}[13][26]\langle 45\rangle^2\langle 6\vert U\vert 2\rangle\langle 3\vert W\vert 2\rangle$ $-t_{345}[13][26]\langle 24\rangle\langle 45\rangle\langle 6\vert U\vert 2\rangle\langle 3\vert V\vert 5\rangle + [12][36]\langle 34\rangle\langle 45\rangle\langle 6\vert U\vert 2\rangle\langle 3\vert V\vert 5\rangle\langle 3\vert W\vert 2\rangle$
$(-,+,-,+)_{(V)}$	$-t_{123}[16]\langle 35\rangle\langle 6\vert V\vert 3\rangle\langle 4\vert V\vert 5\rangle\langle 4\vert W\vert 2\rangle - t_{234}[46]\langle 35\rangle\langle 6\vert V\vert 3\rangle\langle 1\vert U\vert 5\rangle\langle 4\vert W\vert 2\rangle$ $+t_{345}[46]\langle 23\rangle\langle 6\vert V\vert 3\rangle\langle 1\vert U\vert 5\rangle\langle 4\vert V\vert 5\rangle - [46][56]\langle 34\rangle\langle 35\rangle\langle 1\vert U\vert 5\rangle\langle 4\vert V\vert 5\rangle\langle 4\vert W\vert 2\rangle$
$(+,-,+,-)_{(VI)}$	$[12][23][15][35]\langle 14\rangle\langle 24\rangle\langle 26\rangle\langle 56\rangle\langle 5\vert U\vert 2\rangle + (S_{12}S_{23} - S_{12}S_{45})[15][35]\langle 24\rangle\langle 46\rangle\langle 3\vert V\vert 6\rangle$ $-S_{23}S_{16}[35]^2\langle 26\rangle\langle 46\rangle\langle 1\vert W\vert 4\rangle - [15]^2[23][34]\langle 12\rangle\langle 16\rangle\langle 24\rangle\langle 46\rangle\langle 1\vert W\vert 4\rangle$ $+S_{23}S_{16}[15][35]\langle 46\rangle^2\langle 3\vert 1+5\vert 2\rangle + S_{12}S_{15}[15][23][35]\langle 24\rangle\langle 26\rangle\langle 46\rangle$